Weak and strong singular solutions of semilinear fractional elliptic equations

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Abstract

Let $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0,1)$ and $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain containing 0. If δ_0 is the Dirac measure at 0 and $k > 0$, we prove that the weakly singular solution u_k of (E_k) $(-\Delta)^\alpha u + u^p = k\delta_0$ in Ω which vanishes in Ω^c, is a classical solution of (E_*) $(-\Delta)^\alpha u + u^p = 0$ in $\Omega \setminus \{0\}$ with the same outer data. When $\frac{2\alpha}{N-2\alpha} \le 1 + \frac{2\alpha}{N}$, $p \in (0, 1 + \frac{2\alpha}{N}]$ we show that the u_k converges to ∞ in whole Ω when $k \to \infty$, while, for $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, the limit of the u_k is a strongly singular solution of (E_*). The same result holds in the case $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$ excepted if $\frac{2\alpha}{N} < p < 1 + \frac{2\alpha}{N}$.

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1 Introduction

Let Ω be a bounded C^2 domain of $\mathbb{R}^N(N \geq 2)$ containing 0, $\alpha \in (0,1)$ and let δ_0 denote the Dirac measure at 0. In this paper, we study the properties of the weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= k\delta_0 \quad \text{in} \quad \Omega \\ u &= 0 \qquad \text{in} \quad \Omega^c, \end{aligned} \tag{1.1}$$

where $k > 0$ and $p \in (0, \frac{N}{N-2\alpha})$ and $(-\Delta)^\alpha$ is the α-fractional Laplacian defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \epsilon] \\ 1, & \text{if} \quad \mathrm{t} > \epsilon. \end{cases}$$

In 1980, Benilan and Brezis (see [2, 1]) studied the case $\alpha = 1$ in equation (1.1) and proved in particular that equation

$$\begin{aligned} -\Delta u + u^q &= k\delta_0 \quad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{1.2}$$

admits a unique solution u_k for $1 < q < N/(N-2)$, while no solution exists when $q \geq N/(N-2)$. Soon after, Brezis and Véron [3] proved that the problem

$$\begin{aligned} -\Delta u + u^q &= 0 \quad \text{in} \quad \Omega \setminus \{0\} \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.3}$$

admits only the zero solution when $q \geq N/(N-2)$. When $1 < q < N/(N-2)$, Véron in [13] obtained the description of the all the possible singular behaviour of the positive solutions of (1.3). In particular he proved that this behaviour is always isotropic (when $(N+1)/(N-1) \leq q < N/(N-2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:
(i) either $u(x) \sim c_N k|x|^{2-N}$ when $x \to 0$ and k can take any positive value; u is said to have a *weak singularity* at 0, and actually $u = u_k$.
(ii) or $u(x) \sim c_{N,q}|x|^{-\frac{2}{q-1}}$ when $x \to 0$ and u has a *strong singularity* at 0, and $u = u_\infty := \lim_{k\to\infty} u_k$.

A large series of papers has been devoted to the extension of semilinear problems involving the Laplacian to problems where the diffusion operator is non-local, the most classical one being the fractional Laplacian, see e.g. [4, 5, 9, 10, 11]. In a recent work, Chen and Véron [7] considered the problem

$$\begin{aligned}(-\Delta)^\alpha u + u^p = 0 \quad &\text{in} \quad \Omega\setminus\{0\}\\ u = 0 \quad &\text{in} \quad \Omega^c,\end{aligned} \tag{1.4}$$

where $1+\frac{2\alpha}{N} < p < p^*_\alpha := \frac{N}{N-2\alpha}$. They proved that (1.4) admits a singular solution u_s which satisfies

$$\lim_{x\to 0} u_s(x)|x|^{\frac{2\alpha}{p-1}} = c_0, \tag{1.5}$$

for some $c_0 > 0$. Moreover u_s is the unique positive solution of (1.4) such that

$$0 < \liminf_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} \le \limsup_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} < \infty. \tag{1.6}$$

In this article we will call *weakly singular solution* a solution u of (1.4) which satisfies $\limsup_{x\to 0}|u(x)||x|^{N-2\alpha} < \infty$ and *strongly singular solution* if $\lim_{x\to 0}|u(x)||x|^{N-2\alpha} = \infty$.

The existence of solutions of (1.1) is a particular case of the more general problem

$$\begin{aligned}(-\Delta)^\alpha u + g(u) = \nu \quad &\text{in} \quad \Omega\\ u = 0 \quad &\text{in} \quad \Omega^c\end{aligned} \tag{1.7}$$

which has been study by Chen and Véron in [8] under the assumption that g is a subcritical nonlinearity, ν being a positive and bounded Radon measure in Ω.

Definition 1.1 *A function u belonging to $L^1(\Omega)$ is a weak solution of (1.7) if $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi]dx = \int_\Omega \xi d\nu \qquad \forall \xi \in \mathbb{X}_\alpha, \tag{1.8}$$

where $\rho(x) := dist(x, \Omega^c)$ and $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le c_1$ for some $c_1 > 0$,

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

According to Theorem 1.1 in [8], problem (1.1) admits a unique weak solution u_k, moreover,

$$\mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[(\mathbb{G}_\alpha[k\delta_0])^p] \le u_k \le \mathbb{G}_\alpha[k\delta_0] \quad \text{in} \quad \Omega, \tag{1.9}$$

where $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_\Omega G_\alpha(x,y)d\nu(y), \qquad \forall\, \nu \in \mathfrak{M}(\Omega, \rho^\alpha), \tag{1.10}$$

with G_α is the Green kernel of $(-\Delta)^\alpha$ in Ω and $\mathfrak{M}(\Omega, \rho^\alpha)$ denotes the space of Radon measures in Ω such that $\int_\Omega \rho^\alpha d \mid \nu \mid < \infty$. By (1.9),

$$\lim_{x\to 0} u_k(x)|x|^{N-2\alpha} = c_{\alpha,N}k. \tag{1.11}$$

for some $c_{\alpha,N} > 0$. From Theorem 1.1 in [8], there holds

$$u_k(x) \le u_{k+1}(x), \qquad \forall x \in \Omega; \tag{1.12}$$

then there exists

$$u_\infty(x) = \lim_{k\to\infty} u_k(x) \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{1.13}$$

and $u_\infty(x) \in \mathbb{R}_+ \cup \{+\infty\}$.

Motivated by these results and in view of the nonlocal character of the fractional Laplacian, in this article we analyse the connection between the solutions of (1.1) and the ones of (1.4). Our main result is the following

Theorem 1.1 *Assume that* $1 + \frac{2\alpha}{N} \ge \frac{2\alpha}{N-2\alpha}$ *and* $p \in (0, p^*_\alpha)$*. Then* u_k *is a classical solution of (1.4). Furthermore,*
(i) *if* $p \in (0, 1 + \frac{2\alpha}{N})$,

$$u_\infty(x) = \infty \qquad \forall x \in \Omega; \tag{1.14}$$

(ii) *if* $p \in (1 + \frac{2\alpha}{N}, p^*_\alpha)$,

$$u_\infty = u_s,$$

where u_s *is the solution of (1.4) satisfying (1.5).*
Moreover, if $1 + \frac{2\alpha}{N} = \frac{2\alpha}{N-2\alpha}$*, (1.14) holds for* $p = 1 + \frac{2\alpha}{N}$.

The result of part (i) indicates that *even if the absorption is superlinear*, the diffusion dominates and there is no strongly singular solution to problem (1.4). On the contrary, part (ii) points out that the absorption dominates the diffusion; the limit function u_s is the least strongly singular solution of (1.4). Comparing Theorem 1.1 with the results for Laplacian case, part (i) with $p \in (0, 1]$ and (ii) are similar as the Laplacian case, but part (i) with $p \in (1, 1+\frac{2\alpha}{N}]$ is totally different from the one in the case $\alpha = 1$. This striking phenomenon comes comes from the fact that the fractional Laplacian is a nonlocal operator, which requires the solution to belong to $L^1(\Omega)$, therefore no local barrier can be constructed if p is too close to 1.

At end, we consider the case where $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$. It occurs when $N = 2$ and $\frac{\sqrt{5}-1}{2} < \alpha < 1$ or $N = 3$ and $\frac{3(\sqrt{5}-1)}{4} < \alpha < 1$. In this situation, we have the following results.

Theorem 1.2 *Assume that* $1+\frac{2\alpha}{N}<\frac{2\alpha}{N-2\alpha}$ *and* $p\in(0,p^*_\alpha)$. *Then* u_k *is a classical solution of (1.4). Furthermore,*
(i) if $p\in(0,\frac{N}{2\alpha})$, *then*

$$u_\infty(x)=\infty \qquad \forall x\in\Omega;$$

(ii) if $p\in(1+\frac{2\alpha}{N},\frac{2\alpha}{N-2\alpha})$, *then* u_∞ *is a classical solution of (1.4) and there exist* $\rho_0>0$ *and* $c_2>0$ *such that*

$$c_2|x|^{-\frac{(N-2\alpha)p}{p-1}}\le u_\infty\le u_s \qquad \forall x\in B_{\rho_0}\setminus\{0\}; \tag{1.15}$$

(iii) if $p=\frac{2\alpha}{N-2\alpha}$, *then* u_∞ *is a classical solution of (1.4) and there exist* $\rho_0>0$ *and* $c_3>0$ *such that*

$$c_3\frac{|x|^{-\frac{(N-2\alpha)p}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}}\le u_\infty\le u_s \qquad \forall x\in B_{\rho_0}\setminus\{0\}; \tag{1.16}$$

(iv) if $p\in(\frac{2\alpha}{N-2\alpha},p^*_\alpha)$, *then*

$$u_\infty=u_s.$$

We remark that $\frac{N}{2\alpha}<1+\frac{2\alpha}{N}$ if $1+\frac{2\alpha}{N}<\frac{2\alpha}{N-2\alpha}$. Therefore Theorem 1.2 does not provide any description of u_∞ in the region

$$\mathcal{U}:=\left\{(\alpha,p)\in(0,1)\times(1,\tfrac{N}{N-2}):\tfrac{N}{2\alpha}<1+\tfrac{2\alpha}{N},\,\tfrac{N}{2\alpha}<p<1+\tfrac{2\alpha}{N}\right\}.$$

Furthermore, in parts (ii) and (iii), we do not obtain that $u_\infty=u_s$, since (1.15) and (1.16) do not provide sharp estimates on u_∞ in order it to belong to the uniqueness class characterized by (1.6).

The paper is organized as follows. In Section 2, we present some some estimates for the Green kernel and comparison principles. In Section 3, we prove that the weak solution of (1.1) is a classical solution of (1.4). Section 4 is devoted to analyze the limit of weakly singular solutions as $k\to\infty$.

2 Preliminaries

The purpose of this section is to recall some known results. We denote by $B_r(x)$ the ball centered at x with radius r and $B_r:=B_r(0)$.

Lemma 2.1 *Assume that* $0<p<p^*_\alpha$, *then there exists* $c_4,c_5,c_6>1$ *such that*
(i) if $p\in(0,\frac{2\alpha}{N-2\alpha})$,

$$\frac{1}{c_4}\le\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p]\le c_4 \quad \text{in} \;\; B_r\setminus\{0\};$$

(ii) if $p = \frac{2\alpha}{N-2\alpha}$,

$$-\frac{1}{c_5}\ln|x| \le \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \le -c_5 \ln|x| \quad \text{in} \quad B_r\setminus\{0\};$$

(iii) if $p \in (\frac{2\alpha}{N-2\alpha}, p^*_\alpha)$,

$$\frac{1}{c_6}|x|^{2\alpha-(N-2\alpha)p} \le \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \le c_6|x|^{2\alpha-(N-2\alpha)p} \quad \text{in} \quad B_r\setminus\{0\},$$

where $r = \frac{1}{4}\min\{1, dist(0,\partial\Omega)\}$ *and* $\mathbb{G}_\alpha$ *is defined by (1.9).*

Proof. The proof follows easily from Chen-Song's estimates of Green functions [9], see [6, Theorem 5.2] for a detailled computation. □

Theorem 2.1 *Assume that* O *is a bounded domain of* $\mathbb{R}^N$ *and* u_1, u_2 *are continuous in* $\bar{O}$ *and satisfy*

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in} \quad O.$$

Moreover, we assume that $u_1 \ge u_2$ *in* O^c. *Then,*
(i) either $u_1 > u_2$ in O,
(ii) or $u_1 \equiv u_2$ a.e. in $\mathbb{R}^N$.

Proof. The proof refers to [5, Theorem 2.3] (see also [4, Theorem 5.2]). □

The following stability result is proved in [5, Theorem 2.2].

Theorem 2.2 *Suppose that* $\mathcal{O}$ *is a bounded* C^2 *domain and* $h : \mathbb{R} \to \mathbb{R}$ *is continuous. Assume* $\{u_n\}$ *is a sequence of functions, uniformly bounded in* $L^1(\mathcal{O}^c, \frac{dy}{1+|y|^{N+2\alpha}})$, *satisfying*

$$(-\Delta)^\alpha u_n + h(u_n) \ge f_n \ (resp\ (-\Delta)^\alpha u_n + h(u_n) \le f_n\ ) \quad in\ \mathcal{O}$$

in the viscosity sense, where the f_n *are continuous in* $\mathcal{O}$. *If there holds*
(i) $u_n \to u$ *locally uniformly in* $\mathcal{O}$,
(ii) $u_n \to u$ *in* $L^1(\mathbb{R}^N, \frac{dy}{1+|y|^{N+2\alpha}})$,
(iii) $f_n \to f$ *locally uniformly in* $\mathcal{O}$,
then

$$(-\Delta)^\alpha u + h(u) \ge f \ (resp\ (-\Delta)^\alpha u + h(u) \le f\ ) \quad in\ \mathcal{O}$$

in the viscosity sense.

3 Regularity

In this section, we prove that any weak solution of (1.1) is a classical solution of (1.4). To this end, we introduce some auxiliary lemma.

Lemma 3.1 *Assume that $w \in C^{2\alpha+\epsilon}(\bar{B}_1)$ with $\epsilon > 0$ satisfies*

$$(-\Delta)^\alpha w = h \quad \text{in} \quad B_1,$$

where $h \in C^1(\bar{B}_1)$. Then for $\beta \in (0, 2\alpha)$, there exists $c_7 > 0$ such that

$$\|w\|_{C^\beta(\bar{B}_{1/4})} \le c_7(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1+|\cdot|)^{-N-2\alpha} w\|_{L^1(\mathbb{R}^N)}). \tag{3.1}$$

Proof. Let $\eta : \mathbb{R}^N \to [0,1]$ be a C^∞ function such that

$$\eta = 1 \quad \text{in} \quad B_{\frac{3}{4}} \quad \text{and} \quad \eta = 0 \quad \text{in} \quad B_1^c.$$

We denote $v = w\eta$, then $v \in C^{2\alpha+\epsilon}(\mathbb{R}^N)$ and for $x \in B_{\frac{1}{2}}$, $\epsilon \in (0, \frac{1}{4})$,

$$\begin{aligned}
(-\Delta)^\alpha_\epsilon v(x) &= -\int_{\mathbb{R}^N \setminus B_\epsilon} \frac{v(x+y) - v(x)}{|y|^{N+2\alpha}} dy \\
&= (-\Delta)^\alpha_\epsilon w(x) + \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy.
\end{aligned}$$

Together with the fact of $\eta(x+y) = 1$ for $y \in B_\epsilon$, we have

$$\int_{\mathbb{R}^N \setminus B_\epsilon} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy = \int_{\mathbb{R}^N} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy =: h_1(x),$$

thus,

$$(-\Delta)^\alpha v = h + h_1 \quad \text{in} \quad B_{\frac{1}{2}}.$$

For $x \in B_{\frac{1}{2}}$ and $z \in \mathbb{R}^N \setminus B_{\frac{3}{4}}$, there holds

$$|z - x| \ge |z| - |x| \ge |z| - \frac{1}{2} \ge \frac{1}{16}(1 + |z|)$$

which implies

$$\begin{aligned}
|h_1(x)| = \left| \int_{\mathbb{R}^N} \frac{(1-\eta(z))w(z)}{|z-x|^{N+2\alpha}} dz \right| &\le \int_{\mathbb{R}^N \setminus B_{\frac{3}{4}}} \frac{|w(z)|}{|z-x|^{N+2\alpha}} dz \\
&\le 16^{N+2\alpha} \int_{\mathbb{R}^N} \frac{|w(z)|}{(1+|z|)^{N+2\alpha}} dz \\
&= 16^{N+2\alpha} \|(1+|\cdot|)^{-N-2\alpha} w\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

By [11, Proposition 2.1.9], for $\beta \in (0, 2\alpha)$, there exists $c_8 > 0$ such that

$$\begin{aligned} \|v\|_{C^\beta(\bar{B}_{1/4})} &\le c_8(\|v\|_{L^\infty(\mathbb{R}^N)} + \|h + h_1\|_{L^\infty(B_{1/2})}) \\ &\le c_8(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|h_1\|_{L^\infty(B_{1/2})}) \\ &\le c_9(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1 + |\cdot|)^{-N-2\alpha} w\|_{L^1(\mathbb{R}^N)}), \end{aligned}$$

where $c_9 = 16^{N+2\alpha} c_8$. Combining with $w = v$ in $B_{\frac{3}{4}}$, we obtain (3.1). $\square$

Theorem 3.1 *Let $\alpha \in (0, 1)$ and $0 < p < p^*_\alpha$, then the weak solution of (1.1) is a classical solution of (1.4).*

Proof. Let u_k be the weak solution of (1.1). By [8, Theorem 1.1], we have

$$0 \le u_k = \mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[u_k^p] \le \mathbb{G}_\alpha[k\delta_0]. \tag{3.2}$$

We observe that $\mathbb{G}_\alpha[k\delta_0] = k\mathbb{G}_\alpha[\delta_0] = kG_\alpha(\cdot, 0)$ is $C^2_{loc}(\Omega \setminus \{0\})$. Denote by O an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ with $r > 0$. Then $\mathbb{G}_\alpha[k\delta_0]$ is uniformly bounded in $\Omega \setminus B_{r/2}$, so is u_k^p by (3.2).

Let $\{g_n\}$ be a sequence nonnegative functions in $C_0^\infty(\mathbb{R}^N)$ such that $g_n \to \delta_0$ in the weak sense of measures and let w_n be the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = kg_n \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{3.3}$$

From [8], we obtain that

$$u_k = \lim_{n\to\infty} w_n \quad \text{a.e. in} \quad \Omega. \tag{3.4}$$

We observe that $0 \le w_n = \mathbb{G}_\alpha[kg_n] - \mathbb{G}_\alpha[w_n^p] \le k\mathbb{G}_\alpha[g_n]$ and $\mathbb{G}_\alpha[g_n]$ converges to $\mathbb{G}_\alpha[\delta_0]$ uniformly in any compact set of $\Omega \setminus \{0\}$ and in $L^1(\Omega)$; then there exists $c_{10} > 0$ independent of n such that

$$\|w_n\|_{L^\infty(\Omega\setminus B_{r/2})} \le c_{10}k \quad \text{and} \quad \|w_n\|_{L^1(\Omega)} \le c_{10}k.$$

By [10, Corollary 2.4] and Lemma 3.1, there exist $\epsilon > 0$, $\beta \in (0, 2\alpha)$ and positive constants $c_{11}, c_{12}, c_{13} > 0$ independent of n and k, such that

$$\begin{aligned} \|w_n\|_{C^{2\alpha+\epsilon}(O)} &\le c_{11}(\|w_n\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|kg_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{C^\beta(\Omega\setminus B_{\frac{3r}{4}})}) \\ &\le c_{12}(\|w_n\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|kg_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{L^1(\Omega)}) \\ &\le c_{13}(k + k^p). \end{aligned}$$

Therefore, together with (3.4) and the Arzela-Ascoli Theorem, it follows that $u_k \in C^{2\alpha+\frac{\epsilon}{2}}(O)$. This implies that u_k is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega \setminus \{0\}$. Therefore, $w_n \to u_k$ and $g_n \to 0$ uniformly in any compact subset of $\Omega \setminus \{0\}$ as $n \to \infty$. We conclude that u_k is a classical solution of (1.4) by Theorem 2.2. $\square$

Corollary 3.1 *Let u_k be the weak solution of (1.1) and O be an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ with $r > 0$. Then there exist $\epsilon > 0$ and $c_{14} > 0$ independent of k such that*

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)} \le c_{14}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}). \tag{3.5}$$

Proof. By Theorem 3.1, u_k is a solution of (1.4). Then the result follows from [10, Corollary 2.4] and Lemma 3.1 since there exist $\epsilon > 0$, $\beta \in (0, 2\alpha)$ and constants $c_{15}, c_{16} > 0$, independent of k, such that

$$\begin{aligned} \|u_k\|_{C^{2\alpha+\epsilon}(O)} &\le c_{15}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{C^\beta(\Omega\setminus B_{\frac{3r}{4}})}) \\ &\le c_{16}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}). \end{aligned}$$

□

Theorem 3.2 *Assume that the weak solutions u_k of (1.1) satisfy*

$$\|u_k\|_{L^1(\Omega)} \le c_{17} \tag{3.6}$$

for some $c_{17} > 0$ independent of k and that for any $r \in (0, dist(0, \partial\Omega))$, there exists $c_{18} > 0$ independent of k such that

$$\|u_k\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} \le c_{18}. \tag{3.7}$$

Then u_∞ is a classical solution of (1.4).

Proof. Let O be an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ for $0 < r < dist(0, \partial\Omega)$. By (3.5), (3.6) and (3.7), there exist $\epsilon > 0$ and $c_{19} > 0$ independent of k such that

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)} \le c_{19}.$$

Together with (1.13) and the Arzela-Ascoli Theorem, it implies that u_∞ belongs to $C^{2\alpha+\frac{\epsilon}{2}}(O)$. Hence u_∞ is $C^{2\alpha+\frac{\epsilon}{2}}$, locally in $\Omega \setminus \{0\}$. Therefore, $w_n \to u_k$ and $g_n \to 0$ uniformly in any compact set of $\Omega \setminus \{0\}$ as $n \to \infty$. Applying Theorem 2.2 we conclude that u_∞ is a classical solution of (1.4).
□

4 The limit of weakly singular solutions

We recall that u_k denotes the weak solution of (1.1) and $d = \min\{1, dist(0, \partial\Omega)\}$.

4.1 The case $p \in (0, 1 + \frac{2\alpha}{N}]$

Proposition 4.1 *Let $p \in (0,1]$, then $\lim_{k\to\infty} u_k(x) = \infty$ for $x \in \Omega$.*

Proof. We observe that $\mathbb{G}_\alpha[\delta_0], \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] > 0$ in Ω. Since by (1.9)

$$u_k \geq k\mathbb{G}_\alpha[\delta_0] - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p],$$

this implies the claim when $p \in (0,1)$, for any $x \in \Omega$. For $p = 1$, $u_k = ku_1$. The proof follows since $u_1 > 0$ in Ω. $\square$

Now we consider the case of $p \in (1, 1 + \frac{2\alpha}{N}]$. Let $\{r_k\} \subset (0, \frac{d}{2}]$ be a strictly decreasing sequence of numbers satisfying $\lim_{k\to\infty} r_k = 0$. Denote by $\{z_k\}$ the sequence of functions defined by

$$z_k(x) = \begin{cases} -d^{-N}, & x \in B_{r_k} \\ |x|^{-N} - d^{-N}, & x \in B^c_{r_k}. \end{cases} \tag{4.1}$$

Lemma 4.1 *Let $\{\rho_k\}$ be a strictly decreasing sequence of numbers such that $\frac{r_k}{\rho_k} < \frac{1}{2}$ and $\lim_{k\to\infty} \frac{r_k}{\rho_k} = 0$. Then*

$$(-\Delta)^\alpha z_k(x) \leq -c_{1,k}|x|^{-N-2\alpha}, \qquad x \in B^c_{\rho_k}$$

where $c_{1,k} = -c_{20}\log(\frac{r_k}{\rho_k})$ with $c_{20} > 0$ independent of k.

Proof. For any $x \in B^c_{\rho_k}$, there holds

$$\begin{aligned}
(-\Delta)^\alpha z_k(x) &= -\frac{1}{2}\int_{\mathbb{R}^N} \frac{z_k(x+y) + z_k(x-y) - 2z_k(x)}{|y|^{N+2\alpha}} dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^N} \frac{|x+y|^{-N}\chi_{B^c_{r_k}(-x)}(y) + |x-y|^{-N}\chi_{B^c_{r_k}(x)}(y) - 2|x|^{-N}}{|y|^{N+2\alpha}} dy \\
&= -\frac{1}{2}|x|^{-N-2\alpha}\int_{\mathbb{R}^N} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz,
\end{aligned}$$

where $\delta(x,z,r_k) = |z+e_x|^{-N}\chi_{B^c_{\frac{r_k}{|x|}}(-e_x)}(z) + |z-e_x|^{-N}\chi_{B^c_{\frac{r_k}{|x|}}(e_x)}(z) - 2$ and $e_x = \frac{x}{|x|}$.

We observe that $\frac{r_k}{|x|} \leq \frac{r_k}{\rho_k} < \frac{1}{2}$ and $|z \pm e_x| \geq 1 - |z| \geq \frac{1}{2}$ for $z \in B_{\frac{1}{2}}$. Then there exists $c_{21} > 0$ such that

$$|\delta(x,z,r_k)| = ||z+e_x|^{-N} + |z-e_x|^{-N} - 2| \leq c_{21}|z|^2.$$

Therefore,

$$\begin{aligned}
\left|\int_{B_{\frac{1}{2}}(0)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz\right| &\leq \int_{B_{\frac{1}{2}}(0)} \frac{|\delta(x,z,r_k)|}{|z|^{N+2\alpha}} dz \\
&\leq c_{21}\int_{B_{\frac{1}{2}}(0)} |z|^{2-N-2\alpha} dz \leq c_{22},
\end{aligned}$$

where $c_{22} > 0$ is independent of k.
When $z \in B_{\frac{1}{2}}(-e_x)$ there holds

$$\begin{aligned}
\int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz &\geq \int_{B^c_{\frac{1}{2}}(-e_x)} \frac{|z+e_x|^{-N}\chi_{B_{\frac{r_k}{|x|}}(-e_x)}(z) - 2}{|z|^{N+2\alpha}} dz \\
&\geq c_{23} \int_{B_{\frac{1}{2}}(0)\setminus B_{\frac{r_k}{|x|}}(0)} (|z|^{-N} - 2) dz \\
&\geq -c_{24} \log(\frac{r_k}{|x|}) \geq -c_{24} \log(\frac{r_k}{\rho_k}),
\end{aligned}$$

where $c_{23}, c_{24} > 0$ are independent of k.
For $z \in B_{\frac{1}{2}}(e_x)$, we have

$$\int_{B_{\frac{1}{2}}(e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz = \int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz.$$

Finally, for $z \in O := \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(-e_x) \cup B_{\frac{1}{2}}(e_x))$, we have

$$|\int_O \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz| \leq c_{25} \int_{B^c_{\frac{1}{2}}(0)} \frac{|z|^{-N}+1}{|z|^{N+2\alpha}} dz \leq c_{26},$$

where $c_{25}, c_{26} > 0$ are independent of k.
Combining these inequalities we obtain that there exists $c_{20} > 0$ independent of k such that

$$(-\Delta)^\alpha z_k(x)|x|^{N+2\alpha} \leq c_{20} \log(\frac{r_k}{\rho_k}) := c_{1,k},$$

which ends the proof. $\square$

Proposition 4.2 *Assume that*

$$\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}, \quad \max\{1, \frac{2\alpha}{N-2\alpha}\} < p < 1 + \frac{2\alpha}{N} \tag{4.2}$$

and z_k is defined by (4.1) with $r_k = k^{-\frac{p-1}{N-(N-2\alpha)p}}(\log k)^{-2}$. Then there exists $k_0 > 0$ such that for any $k \geq k_0$

$$u_k \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in} \quad B_d, \tag{4.3}$$

where $c_{2,k} = \ln\ln k$.

Proof. For $p \in (\max\{1, \frac{2\alpha}{N-2\alpha}\}, 1+\frac{2\alpha}{N})$, it follows by (1.9) and Lemma 2.1-(iii) that there exist $\rho_0 \in (0,d)$ and $c_{27}, c_{28} > 0$ independent of k such that, for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$\begin{aligned} u_k(x) &\geq k\mathbb{G}_\alpha[\delta_0](x) - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) \\ &\geq c_{27}k|x|^{-N+2\alpha} - c_{28}k^p|x|^{-(N-2\alpha)p+2\alpha} \\ &= c_{27}k|x|^{-N+2\alpha}(1-\frac{c_{28}}{c_{27}}k^{p-1}|x|^{N-(N-2\alpha)p}). \end{aligned}$$

We choose

$$\rho_k = k^{-\frac{p-1}{N-(N-2\alpha)p}}(\log k)^{-1}. \tag{4.4}$$

There exits $k_1 > 1$ such that for $k \geq k_1$

$$\begin{aligned} u_k(x) &\geq c_{27}k|x|^{-N+2\alpha}(1-\frac{c_{28}}{c_{27}}k^{p-1}\rho_k^{N-(N-2\alpha)p}) \\ &\geq \frac{c_{27}}{2}k|x|^{-N+2\alpha}, \quad x \in \bar{B}_{\rho_k} \setminus \{0\}. \end{aligned} \tag{4.5}$$

Since $p < 1 + \frac{2\alpha}{N}$, $1 - \frac{2\alpha(p-1)}{N-(N-2\alpha)p} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{27}}{2}kr_k^{2\alpha} \geq (\ln\ln k)^{\frac{1}{p-1}}, \tag{4.6}$$

for $k \geq k_0$. This implies

$$\frac{c_{27}}{2}k|x|^{2\alpha} \geq (\ln\ln k)^{\frac{1}{p-1}}, \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k}.$$

Together with (4.1) and (4.5), we derive

$$u_k(x) \geq (\ln\ln k)^{\frac{1}{p-1}} z_k(x), \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k},$$

for $k \geq k_0$. Furthermore, it is clear that

$$(\ln\ln k)^{\frac{1}{p-1}} z_k(x) \leq 0 \leq u_k(x)$$

whenever $x \in B_{r_k}$ or $x \in B_d^c$. Set $c_{2,k} = \ln\ln k$, then by Lemma 4.1

$$(-\Delta)^\alpha c_{2,k}^{\frac{1}{p-1}} z_k(x) + c_{2,k}^{\frac{p}{p-1}} z_k(x)^p \leq c_{2,k}^{\frac{p}{p-1}} |x|^{-N-2\alpha}(-1+|x|^{N+2\alpha-Np}) \leq 0,$$

for any $x \in B_d \setminus B_{\rho_k}$, since $N+2\alpha-Np \geq 0$ and $d \leq 1$. Applying Theorem 2.1, we infer that

$$c_{2,k}^{\frac{1}{p-1}} z_k(x) \leq u_k(x) \qquad \forall x \in \bar{B}_d,$$

which ends the proof. $\square$

Proposition 4.3 *Assume*

$$1 < \frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N} \quad \text{and} \quad p = \frac{2\alpha}{N-2\alpha} \tag{4.7}$$

and let z_k be defined by (4.1) with $r_k = k^{-\frac{2\alpha}{N(N-2\alpha)}}(\log k)^{-3}$ and $k > 2$. Then there exists $k_0 > 2$ such that (4.3) holds for $k \geq k_0$.

Proof. By (1.9) and Lemma 2.1-(ii), there exist $\rho_0 \in (0, d)$ and $c_{30}, c_{31} > 0$ independent of k, such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$

$$\begin{aligned} u_k(x) &\geq c_{30} k|x|^{-N+2\alpha} + c_{31} k^p \log|x| \\ &= c_{30} k|x|^{-N+2\alpha}\left(1 + \frac{c_{31}}{c_{30}} k^{p-1}|x|^{N-2\alpha}\log|x|\right). \end{aligned}$$

If we choose $\rho_k = k^{-\frac{2\alpha}{N(N-2\alpha)}}(\log k)^{-2}$ there exists $k_1 > 1$ such that for $k \geq k_1$, we have $1 + \frac{c_{31}}{c_{30}} k^{p-1}\rho_k^{N-2\alpha}\log(\rho_k) \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{30}}{2} k|x|^{-N+2\alpha} \qquad \forall x \in \bar{B}_{\rho_k} \setminus \{0\}. \tag{4.8}$$

Since $\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$, there holds $1 - \frac{4\alpha^2}{N(N-2\alpha)} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{30}}{2} k r_k^{2\alpha} = \frac{c_{30}}{2} k^{1-\frac{4\alpha^2}{N(N-2\alpha)}}(\log k)^{-6\alpha} \geq (\ln\ln k)^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is the same as in Proposition 4.2. $\square$

In the sequel, we point out the fact that the limit behavior of the u_k depends which of the following three cases holds:

$$\frac{2\alpha}{N-2\alpha} = 1 + \frac{2\alpha}{N} = \frac{N}{2\alpha}; \tag{4.9}$$

$$\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N} < \frac{N}{2\alpha}; \tag{4.10}$$

$$\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N} > \frac{N}{2\alpha}. \tag{4.11}$$

Proposition 4.4 *Assume*

$$1 < \frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N} \quad \text{and} \quad 1 < p < \frac{2\alpha}{N-2\alpha}, \tag{4.12}$$

or

$$1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha} \quad \text{and} \quad 1 < p < \frac{N}{2\alpha}, \tag{4.13}$$

and z_k is defined by (4.1) with $r_k = k^{-\frac{p-1}{N-2\alpha}}(\log k)^{-1}$. Then there exists $k_0 > 2$ such that (4.3) holds for $k \geq k_0$.

Proof. By (1.9) and Lemma 2.1-(i), there exist $\rho_0 \in (0, d)$ and $c_{33}, c_{34} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$\begin{aligned} u_k(x) &\geq c_{33} k|x|^{-N+2\alpha} - c_{34} k^p \\ &= c_{33} k|x|^{-N+2\alpha}(1 - \frac{c_{34}}{c_{33}} k^{p-1} |x|^{N-2\alpha}). \end{aligned}$$

We choose $\rho_k = k^{-\frac{p-1}{N-2\alpha}}$. Then there exists $k_1 > 1$ such that for $k \geq k_1$, $1 - \frac{c_{34}}{c_{33}} k^{p-1} \rho_k^{N-2\alpha} \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{33}}{2} k|x|^{-N+2\alpha} \qquad \forall x \in \bar{B}_{\rho_k} \setminus \{0\}. \tag{4.14}$$

Clearly $p < \frac{N}{2\alpha}$ by assumptions (4.12), (4.13), together with relations (4.9)(4.10), (4.11), thus $1 - (p-1)\frac{2\alpha}{N-2\alpha} > 0$. Therefore there exists $k_0 \geq k_1$ such that

$$\frac{c_{33}}{2} k r_k^{2\alpha} = \frac{c_{33}}{2} k^{1-(p-1)\frac{2\alpha}{N-2\alpha}} (\log k)^{-2\alpha} \geq (\log \log k)^{\frac{1}{p-1}} = c_{2,k}^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is similar to the one of Proposition 4.2. $\square$

4.2 The case $p \in (1, 1 + \frac{2\alpha}{N}]$

We give below the proof, in two steps, of Theorem 1.1 part (i) with $p \in (1, 1 + \frac{2\alpha}{N}]$ and Theorem 1.2 part (i) with $p \in (1, \frac{2\alpha}{N}]$.

Step 1: We claim that $u_\infty = \infty$ in B_d. We observe that for $\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$, Propositions 4.2, 4.3, 4.4 cover the case $p \in (\max\{1, \frac{2\alpha}{N-2\alpha}\}, 1+\frac{2\alpha}{N})$, the case $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p = \frac{2\alpha}{N-2\alpha}$ and the case $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p \in (1, \frac{2\alpha}{N-2\alpha})$ respectively. For $\frac{2\alpha}{N-2\alpha} = 1 + \frac{2\alpha}{N}$, Proposition 4.3, 4.4 cover the case $p = \frac{2\alpha}{N-2\alpha}$ and the case $p \in (1, \frac{2\alpha}{N-2\alpha})$ respectively. So it covers $p \in (1, 1 + \frac{2\alpha}{N}]$ in Theorem 1.1 part (i). When $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$, Proposition 4.4 covers $p \in (1, \frac{N}{2\alpha})$ in Theorem 1.2 part (i). Therefore, we have

$$u_\infty \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in} \quad B_d$$

and since for any $x \in B_d \setminus \{0\}$, $\lim_{k\to\infty} c_{2,k}^{\frac{1}{p-1}} z_k(x) = \infty$, we deive

$$u_\infty = \infty \quad \text{in} \quad B_d.$$

Step 2: We claim that $u_\infty = \infty$ in Ω. By the fact of $u_\infty = \infty$ in B_d and $u_{k+1} \geq u_k$ in Ω, then for any $n > 1$ there exists $k_n > 0$ such that $u_{k_n} \geq n$ in B_d. For any $x_0 \in \Omega \setminus B_d$, there exists $\rho > 0$ such that $\bar{B}_\rho(x_0) \subset \Omega \cap B^c_{d/2}$. We denote by w_n the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = 0 \quad &\text{in} \quad B_\rho(x_0) \\ u = 0 \quad &\text{in} \quad B^c_\rho(x_0) \setminus B_{d/2} \\ u = n \quad &\text{in} \quad B_{d/2}. \end{aligned} \tag{4.15}$$

Then by Theorem 2.1, we have

$$u_{k_n} \geq w_n. \tag{4.16}$$

Let η_1 be the solution of

$$\begin{aligned} (-\Delta)^\alpha u = 1 \quad &\text{in} \quad B_\rho(x_0) \\ u = 0 \quad &\text{in} \quad B^c_\rho(x_0), \end{aligned}$$

and $v_n = w_n - n\chi_{B_{d/2}}$, then $v_n = w_n$ in $B_\rho(x_0)$ and

$$\begin{aligned} (-\Delta)^\alpha v_n(x) + v_n^p(x) &= (-\Delta)^\alpha w_n(x) - n(-\Delta)^\alpha \chi_{B_{d/2}}(x) + w_n^p(x) \\ &= n\int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}} \qquad \forall x \in B_\rho(x_0). \end{aligned}$$

This means that v_n is a solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = n\int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}} \quad &\text{in} \quad B_\rho(x_0), \\ u = 0 \quad &\text{in} \quad B^c_\rho(x_0). \end{aligned} \tag{4.17}$$

It is clear that

$$\frac{1}{c_{35}} \leq \int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}} \leq c_{35} \qquad \forall x \in B_\rho(x_0)$$

for some $c_{35} > 1$. Furhermore $(\frac{n}{2c_{35}\max\eta_1})^{\frac{1}{p}}\eta_1$ is sub solution of (4.17) for n large enough. Then using Theorem 2.1, we obtain that

$$v_n \geq (\frac{n}{2c_{35}\max\eta_1})^{\frac{1}{p}}\eta_1 \qquad \forall x \in B_\rho(x_0),$$

which implies that

$$w_n \geq (\frac{n}{2c_{35}\max\eta_1})^{\frac{1}{p}}\eta_1 \qquad \forall x \in B_\rho(x_0).$$

Then

$$\lim_{n\to\infty} w_n(x_0) \to \infty.$$

Since x_0 is arbitrary and together with (4.16), it implies that $u_\infty = \infty$ in Ω, which completes the proof. $\square$

4.3 The case of $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$

Proposition 4.5 *Let $\alpha \in (0,1)$ and $r_0 = dist(0, \partial\Omega)$. Then*
(i) if $\max\{1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha}\} < p < p^*_\alpha$, *there exist $R_0 \in (0, r_0)$ and $c_{36} > 0$ such that*

$$u_\infty(x) \geq c_{36}|x|^{-\frac{2\alpha}{p-1}} \qquad \forall x \in B_{R_0} \setminus \{0\}, \tag{4.18}$$

(ii) if $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$ *and* $p = \frac{2\alpha}{N-2\alpha}$*, there exist* $R_0 \in (0, r_0)$ *and* $c_{37} > 0$ *such that*

$$u_\infty(x) \geq \frac{c_{37}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}} |x|^{-\frac{p(N-2\alpha)}{p-1}}, \quad \forall x \in B_{R_0} \setminus \{0\}, \tag{4.19}$$

(iii) if $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$ *and* $p \in (1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$*, there exist* $R_0 \in (0, r_0)$ *and* $c_{38} > 0$ *such that*

$$u_\infty(x) \geq c_{38} |x|^{-\frac{p(N-2\alpha)}{p-1}} \qquad \forall x \in B_{R_0} \setminus \{0\}. \tag{4.20}$$

Proof. (i) Using (1.9) and Lemma 2.1(i) with $\max\{1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha}\} < p < p^*_\alpha$, we see that there exist $\rho_0 \in (0, r_0)$ and $c_{39}, c_{40} > 0$ such that

$$u_k(x) \geq c_{39} k |x|^{-N+2\alpha} - c_{40} k^p |x|^{-(N-2\alpha)p+2\alpha} \qquad \forall x \in B_{\rho_0} \setminus \{0\}. \tag{4.21}$$

Set

$$\rho_k = (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}} k^{p-1})^{\frac{1}{(N-2\alpha)(p-1)-2\alpha}}. \tag{4.22}$$

Since $(N-2\alpha)(p-1) - 2\alpha < 0$, there holds $\lim_{k\to\infty} \rho_k = 0$. Let $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$, then for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we have

$$\begin{aligned} c_{40} k^p |x|^{-(N-2\alpha)p+2\alpha} &\leq c_{40} k^p (\frac{\rho_k}{2})^{-(N-2\alpha)p+2\alpha} \\ &= \frac{c_{39}}{2} k \rho_k^{-N+2\alpha} \\ &\leq \frac{c_{39}}{2} k |x|^{-N+2\alpha} \end{aligned}$$

and

$$k = (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}})^{-\frac{1}{p-1}} \rho_k^{N-2\alpha-\frac{2\alpha}{p-1}} \geq c_{41} |x|^{N-2\alpha-\frac{2\alpha}{p-1}},$$

where $c_{41} = (2^{(N-2\alpha)p-2\alpha-1} \frac{c_{40}}{c_{39}})^{-\frac{1}{p-1}} 2^{(N-2\alpha)(p-1)-2\alpha-1}$. Combining with (4.18), we obtain

$$\begin{aligned} u_k(x) &= c_{39} k |x|^{-N+2\alpha} - c_{40} k^p |x|^{-(N-2\alpha)p+2\alpha} \\ &\geq \frac{c_{39}}{2} k |x|^{-N+2\alpha} \\ &\geq c_{42} |x|^{-\frac{2\alpha}{p-1}}, \end{aligned} \tag{4.23}$$

for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, where $c_{42} = \frac{1}{2} c_{39} c_{41}$ is independent of k. By (4.22), we can choose a sequence $\{k_n\} \subset [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2} \rho_{k_n},$$

For any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$, then, by (4.23),

$$u_{k_n}(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}.$$

Together with $u_{k+1} > u_k$, we derive

$$u_\infty(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}, \quad x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(ii) By (1.9) and Lemma 2.1-(ii) with $p = \frac{2\alpha}{N-2\alpha}$, there exist $\rho_0 \in (0, r_0)$ and $c_{43}, c_{44} > 0$ such that

$$u_k(x) \geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\log(|x|)|, \quad x \in B_{\rho_0} \setminus \{0\}. \tag{4.24}$$

Let $\{\rho_k\}$ be a sequence of real numbers with value in $(0,1)$ and such that

$$c_{44}k^{p-1}|\log(\frac{\rho_k}{2})| = \frac{c_{43}}{2}\rho_k^{-N+2\alpha}. \tag{4.25}$$

Then $\lim_{k\to\infty} \rho_k = 0$ and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Thus, for any $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$,

$$c_{43}k^p|\log(|x|)| \leq c_{44}k^p|\log(\frac{\rho_k}{2})| = \frac{c_{43}}{2}k\rho_k^{-N+2\alpha} \leq \frac{c_{43}}{2}k|x|^{-N+2\alpha}.$$

Therefore, assuming always $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we derive from (4.25) that

$$k = (\frac{c_{44}}{2c_{43}})^{-\frac{1}{p-1}}(\frac{\rho_k^{-N+2\alpha}}{1+|\log(\rho_k)|})^{\frac{1}{p-1}} \geq c_{45}\frac{|x|^{-\frac{N-2\alpha}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}},$$

where $c_{45} = 2^{-\frac{N-2\alpha}{p-1}}(\frac{c_{44}}{2c_{43}})^{-\frac{1}{p-1}}$. Consequently

$$\begin{aligned} u_k(x) &\geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\log(|x|)| \\ &\geq \frac{c_{43}}{2}k|x|^{-N+2\alpha} \geq c_{46}\frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}}, \end{aligned} \tag{4.26}$$

where $c_{46} = \frac{1}{2}c_{43}c_{45}$ is independent of k.

By (4.25), we can choose a sequence $k_n \in [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$. By (4.26) there holds

$$u_{k_n}(x) \geq c_{46}\frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}}.$$

Together with $u_{k+1} > u_k$, we infer

$$u_\infty(x) \geq c_{46} \frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1 + |\log(|x|)|)^{\frac{1}{p-1}}} \qquad \forall x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(iii) By (1.9) and Lemma 2.1-(iii) with $p \in (1 + \frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, there exist $\rho_0 \in (0, r_0)$ and $c_{47}, c_{48} > 0$ such that

$$u_k(x) \geq c_{47} k|x|^{-N+2\alpha} - c_{48} k^p \qquad \forall x \in B_{\rho_0} \setminus \{0\}. \tag{4.27}$$

Put

$$\rho_k = (\frac{c_{48}}{2c_{47}} k^{p-1})^{-\frac{1}{N-2\alpha}}, \tag{4.28}$$

then $\lim_{k\to\infty} \rho_k = 0$ and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Therefore, if $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$, there holds

$$c_{48} k^p = \frac{c_{47}}{2} k \rho_k^{-N+2\alpha} \leq \frac{c_{47}}{2} k|x|^{-N+2\alpha},$$

which yields

$$k = (\frac{c_{48}}{2c_{47}})^{-\frac{1}{p-1}} \rho_k^{-\frac{N-2\alpha}{p-1}} \geq c_{49}|x|^{-\frac{N-2\alpha}{p-1}},$$

by (4.28), where $c_{49} = 2^{-\frac{N-2\alpha}{p-1}} (\frac{c_{48}}{2c_{47}})^{-\frac{1}{p-1}}$. Consequently,

$$\begin{aligned} u_k(x) \geq c_{47} k|x|^{-N+2\alpha} - c_{48} k^p &\geq \frac{c_{47}}{2} k|x|^{-N+2\alpha} \\ &\geq c_{50}|x|^{-\frac{p}{p-1}(N-2\alpha)}, \end{aligned} \tag{4.29}$$

where $c_{50} = \frac{1}{2} c_{47} c_{49}$ is independent of k.

By (4.28), we can choose a sequence $k_n \in [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2} \rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$ and then by (4.29),

$$u_{k_n}(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}}.$$

Together with $u_{k+1} > u_k$, we have

$$u_\infty(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}} \qquad \forall x \in B_{\rho_{k_0}} \setminus \{0\},$$

which ends the proof. $\square$

Lemma 4.2 *Let $p \in (1+\frac{2\alpha}{N}, p_\alpha^*)$ and u_s be a strongly singular solution of (1.4) satisfying (1.5). Then*

$$u_\infty \le u_s \quad \text{in} \quad \Omega \setminus \{0\}, \tag{4.30}$$

where u_∞ is defined by (1.13).

Proof. By (1.5) and (1.9), it follows

$$\lim_{x\to 0} u_s |x|^{\frac{2\alpha}{p-1}} = c_0 \quad \text{and} \quad \lim_{x\to 0} u_k |x|^{N-2\alpha} = c_k,$$

which implies that there exists $r_1 > 0$ such that

$$u_k < u_s \quad \text{in} \quad B_{r_1} \setminus \{0\}.$$

Since by Theorem 3.1, u_k satisfies

$$(-\Delta)^\alpha u_k + u_k^p = 0 \quad \text{in} \quad \Omega \setminus B_{r_1}(0),$$

so does u_s. By Theorem 2.1 there holds $u_k \le u_s$ in $\Omega \setminus \{0\}$. Jointly with (1.13), it implies

$$u_\infty \le u_s \quad \text{in} \quad \Omega \setminus \{0\}.$$

$\square$

Proof of Theorem 1.1 (ii) **and Theorem 1.2** (iv)**.** By Lemma 4.2 and Theorem 3.2, we obtain that u_∞ is a classical solution of (1.4). Moreover, by Proposition 4.5 part (i) and Lemma 4.2, we have

$$\frac{1}{c_{51}} |x|^{-\frac{2\alpha}{p-1}} \le u_\infty(x) \le c_{51} |x|^{-\frac{2\alpha}{p-1}},$$

for some $c_{51} > 1$. Then $u_\infty = u_s$ in $\mathbb{R}^N \setminus \{0\}$ since u_s is unique in the class of solutions satisfying (1.6). $\square$

4.4 Proof of Theorem 1.2 (ii) and (iii)

By Lemma 4.2 and Theorem 3.2, u_∞ is a classical solution of (1.4) and it satisfies

$$u_\infty \le u_s \quad \text{in} \quad \Omega \setminus \{0\}.$$

Therefore (1.16) and (1.15) follow by Proposition 4.5 part (ii) and (iii), respectively.

$\square$

References

[1] Ph. Benilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, *J. Evolution Eq. 3*, 673-770, (2003).

[2] H. Brezis, Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems, *Proc. Internat. School, Erice, Wiley, Chichester*, 53-73 (1980).

[3] H. Brezis and L. Véron, Removable singularities of some nonlinear elliptic equations, *Arch. Rational Mech. Anal. 75*, 1-6 (1980).

[4] L. Caffarelli and L. Silvestre, Regularity theory for fully non-linear integrodifferential equations, *Comm. Pure Appl. Math. 62*, 597-638 (2009).

[5] H. Chen, P. Felmer and A. Quaas, Large solutions to elliptic equations involving fractional Laplacian, *submitted.*

[6] H. Chen and L. Véron, Solutions of fractional equations involving sources and Radon measures, *HAL : hal-00766824, version 1, Dec. 2012.*

[7] H. Chen and L. Véron, Singular solutions of fractional elliptic equations with absorption, *arXiv:1302.1427v1, [math.AP]*, 6 (Feb. 2013).

[8] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *arXiv:1305.0945v2, [math.AP]*, 15 (May 2013).

[9] Z. Chen, and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann. 312*, 465-501 (1998).

[10] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl.*, to appear.

[11] L. Silvestre, Regularity of the obstacle problem for a fractional power of the laplace operator, *Comm. Pure Appl. Math. 60,* 67-112 (2007).

[12] L. Véron, Weak and strong singularities of nonlinear elliptic equations, *Proc. Symp. Pure Math. 45,* 477-495 (1986).

[13] L. Véron, Singular solutions of some nonlinear elliptic equations, *Nonlinear Anal. T. M. & A. 5,* 225-242 (1981).

[14] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, *Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam* (2004).